Model Solutions

Please check the examination details below before entering your candidate information

Candidate surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number

Candidate Number

Monday 11 May 2020

Morning (Time: 1 hour 30 minutes) | Paper Reference **9FM0/01**

Further Mathematics

Advanced

Paper 1: Core Pure Mathematics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided
  – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 7 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets
  – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

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1. $$f(z) = 3z^3 + pz^2 + 57z + q$$

where p and q are real constants.

Given that $3 - 2\sqrt{2}\mathrm{i}$ is a root of the equation $f(z) = 0$

(a) show all the roots of $f(z) = 0$ on a single Argand diagram,

(7)

(b) find the value of p and the value of q.

(3)

As $3-2i\sqrt{2}$ is a root, then $3+2i\sqrt{2}$ is also a root as it is the conjugate pair of $3-2i\sqrt{2}$.

method 1 to find the third root

$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a} = \frac{57}{3} = 19$

$(3-2i\sqrt{2})(3+2i\sqrt{2}) + (3+2i\sqrt{2})\gamma + (3-2i\sqrt{2})\gamma = 19$

$9 + 8 + 3\gamma + 2i\gamma\sqrt{2} + 3\gamma - 2i\gamma\sqrt{2} = 19$

$17 + 6\gamma = 19$

$6\gamma = 2$

$\gamma = \frac{1}{3}$

method 2 to find the third root:

$(z-3-2i\sqrt{2})(z-3+2i\sqrt{2})$

$= z^2 - 6z + 9 + 8$

$= z^2 - 6z + 17$

$(z^2-6z+17)(az+b) = 3z^3 + pz^2 + 57z + q$

compare coefficients:

z^3 coefficients : $a = 3$

z coefficients : $-6b + 17a = 57$

$-6b + 17(3) = 57$

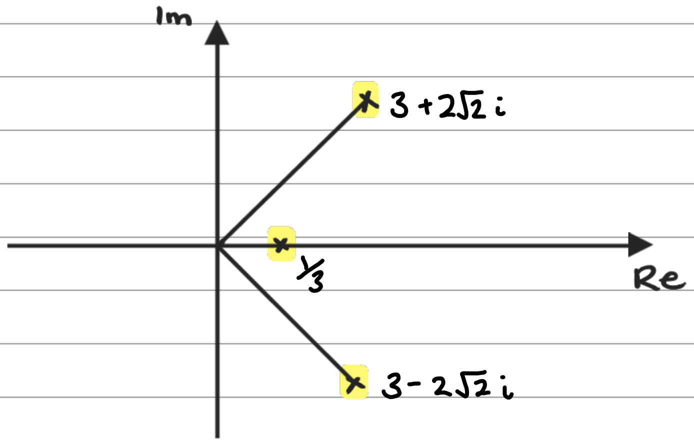
$-6b + 51 = 57$

$-6b = 6$

$b = -1$

$(3z-1) = 0$ $\therefore$ third root is $\frac{1}{3}$

Question 1 continued



b) method 1

$\alpha+\beta+\gamma = -\frac{b}{a} = -\frac{p}{3}$

$(3+2i\sqrt{2}) + (3-2i\sqrt{2}) + \frac{1}{3} = -\frac{p}{3}$

$\Rightarrow \frac{19}{3} = -\frac{p}{3}$

$\Rightarrow -19 = p$

$\alpha\beta\gamma = \frac{-d}{a} = \frac{-q}{3}$

$\frac{1}{3}(3+2i\sqrt{2})(3-2i\sqrt{2}) = -\frac{q}{3}$

$\Rightarrow \frac{1}{3}(9+8) = -\frac{q}{3}$

$9+8 = -q$

$-17 = q$

method 2

$(z^2-6z+17)(3z-1) = 0$

$3z^3 - z^2 - 18z^2 + 6z + 51z - 17 = 0$

$3z^3 - 19z^2 + 57z - 17 = 0$

$\therefore$ $p = -19$, $q = -17$

Turn over

Question 1 continued

Question 1 continued

(Total for Question 1 is 10 marks)

2. (a) Explain why $\int_1^{\infty} \frac{1}{x(2x+5)}\,dx$ is an improper integral.

(1)

(b) Prove that

$$\int_1^{\infty} \frac{1}{x(2x+5)}\,dx = a\ln b$$

where a and b are rational numbers to be determined.

(6)

a) · As the interval being integrated over is unbounded (undefined).
• As the upper limit is infinity.
• As a limit is required to evaluate it.

b) Partial Fractions:

$$\frac{1}{x(2x+5)} = \frac{A}{x} + \frac{B}{2x+5}$$

$$1 = A(2x+5) + Bx$$

compare coefficients:

x coefficients: $2A + B = 0$

constants: $5A = 1$

$A = \frac{1}{5}$ $\quad B = -\frac{2}{5}$

$$\frac{1}{x(2x+5)} = \frac{1}{5x} - \frac{2}{5(2x+5)}$$

$$\int_1^{\infty} \frac{1}{5x} - \frac{2}{5(2x+5)}\,dx$$

$$= \frac{1}{5}\int_1^{\infty} x^{-1} - 2(2x+5)^{-1}\,dx$$

$$\therefore \lim_{t\to\infty} \frac{1}{5}\int_1^{t} x^{-1} - 2(2x+5)^{-1}\,dx$$

Question 2 continued

$$\lim_{t\to\infty} \frac{1}{5}\left[\ln x - \ln(2x+5)\right]_1^t$$

$$= \lim_{t\to\infty} \frac{1}{5}\left(\left(\ln t - \ln(2t+5)\right) + \ln 7\right)$$

$$= \lim_{t\to\infty} \frac{1}{5}\left(\ln\left(\frac{t}{2t+5}\right) + \ln 7\right)$$

As $t \to \infty$, $\ln\left(\frac{t}{2t+5}\right) \to \ln\frac{1}{2}$

$$\therefore \int_1^\infty \frac{1}{x(2x+5)}\,dx = \frac{1}{5}\left(\ln\frac{1}{2} + \ln 7\right)$$

$$= \frac{1}{5}\ln\frac{7}{2}$$

(Total for Question 2 is 7 marks)

3.

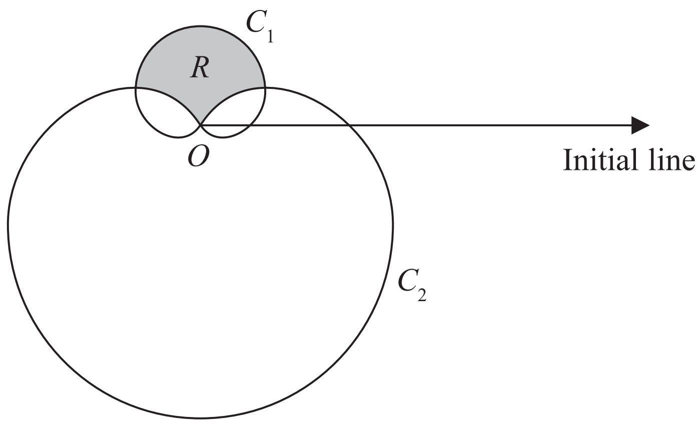


Figure 1

Figure 1 shows a sketch of two curves C_1 and C_2 with polar equations

$$C_1: r = (1 + \sin\theta) \qquad 0 \leqslant \theta < 2\pi$$

$$C_2: r = 3(1 - \sin\theta) \qquad 0 \leqslant \theta < 2\pi$$

The region R lies inside C_1 and outside C_2 and is shown shaded in Figure 1.

Show that the area of R is

$$p\sqrt{3} - q\pi$$

where p and q are integers to be determined.

(9)

when $C_1 = C_2$:

$1 + \sin\theta = 3(1 - \sin\theta)$

$1 + \sin\theta = 3 - 3\sin\theta$

$4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$ (or $\frac{5\pi}{6}$)

Area of C_1 between $\frac{\pi}{6}$ and $\frac{5\pi}{6}$:

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1 + \sin\theta)^2 \, d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 1 + 2\sin\theta + \sin^2\theta \, d\theta$$

As $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$$\therefore \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 1 + 2\sin\theta + \frac{1}{2}(1 - \cos 2\theta) \, d\theta$$

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Question 3 continued

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos 2\theta \, d\theta$$

$$= \frac{1}{2}\left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= \frac{1}{2}\left(\left(\frac{5\pi}{4} + \sqrt{3} + \frac{\sqrt{3}}{8}\right) - \left(\frac{\pi}{4} - \sqrt{3} - \frac{\sqrt{3}}{8}\right)\right)$$

$$= \frac{1}{2}\left(\pi + \frac{9\sqrt{3}}{4}\right) = \frac{\pi}{2} + \frac{9\sqrt{3}}{8}$$

Area of C_2 between $\frac{\pi}{6}$ and $\frac{5\pi}{6}$:

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(3(1-\sin\theta)\right)^2 = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 9\left(1 - 2\sin\theta + \sin^2\theta\right) \, d\theta$$

As $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$$= \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

$$= \frac{9}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 1 - 2\sin\theta + \frac{1}{2} - \frac{1}{2}\cos 2\theta \, d\theta$$

$$= \frac{9}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{3}{2} - 2\sin\theta - \frac{1}{2}\cos 2\theta \, d\theta$$

$$= \frac{9}{2}\left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= \frac{9}{2}\left(\left(\frac{5\pi}{4} - \sqrt{3} + \frac{\sqrt{3}}{8}\right) - \left(\frac{\pi}{4} + \sqrt{3} - \frac{\sqrt{3}}{8}\right)\right)$$

Question 3 continued

$$= \frac{9}{2}\left(\pi - \frac{7\sqrt{3}}{4}\right)$$

$$= \frac{9\pi}{2} - \frac{63\sqrt{3}}{8}$$

Area of R:

$$\left(\frac{\pi}{2} + \frac{9\sqrt{3}}{8}\right) - \left(\frac{9\pi}{2} - \frac{63\sqrt{3}}{8}\right)$$

$$= 9\sqrt{3} - 4\pi$$

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Question 3 continued

(Total for Question 3 is 9 marks)

4. The plane Π_1 has equation

$$\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + \mu(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

where λ and μ are scalar parameters.

(a) Find a Cartesian equation for Π_1

(4)

The line l has equation

$$\frac{x-1}{5} = \frac{y-3}{-3} = \frac{z+2}{4}$$

(b) Find the coordinates of the point of intersection of l with Π_1

(3)

The plane Π_2 has equation

$$\mathbf{r}.(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 5$$

(c) Find, to the nearest degree, the acute angle between Π_1 and Π_2

(2)

a) Using the cross product:

$$\begin{vmatrix} i & j & k \\ 1 & 2 & -3 \\ -1 & 2 & 1 \end{vmatrix} = \begin{pmatrix} (2\times1)-(-3\times2) \\ -(1-(-1\times-3)) \\ (2\times1)-(-1\times2) \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ 2 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 2 \\ 4 \end{pmatrix} = 16+8-4 = 20$$

Cartesian Equation of Π:

$8x+2y+4z=20$

$4x+y+2z=10$

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Question 4 continued

b) Line equation:

$$\underline{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + t \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix}$$

Sub the line equation into the cartesian equation of Π:

$$4(1+5t) + (3-3t) + 2(-2+4t) = 10$$
$$4 + 20t + 3 - 3t - 4 + 8t = 10$$
$$25t = 7$$
$$t = \frac{7}{25}$$

Sub $t = 7/25$ into the line equation:

$$\begin{pmatrix} 1 + (7/25)5 \\ 3 - (7/25)3 \\ -2 + (7/25)4 \end{pmatrix} = \begin{pmatrix} 12/5 \\ 54/25 \\ -22/25 \end{pmatrix}$$

$\therefore$ coordinates $\left(12/5, \frac{54}{25}, \frac{-22}{25}\right)$

Question 4 continued

c) Using equation $\cos^{-1}\left(\frac{a.b}{|a||b|}\right) = \theta$

$a.b = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 8 - 1 + 6 = 13$

$|a| = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21}$

$|b| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$

$\cos^{-1}\left(\frac{13}{\sqrt{21} \times \sqrt{14}}\right) = 40.69639215°$

$= 41°$

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P 6 2 6 7 2 A 0 1 4 2 8

Question 4 continued

(Total for Question 4 is 9 marks)

5. Two compounds, X and Y, are involved in a chemical reaction. The amounts in grams of these compounds, t minutes after the reaction starts, are x and y respectively and are modelled by the differential equations

$$\frac{dx}{dt} = -5x + 10y - 30$$

$$\frac{dy}{dt} = -2x + 3y - 4$$

(a) Show that

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 50$$

(3)

(b) Find, according to the model, a general solution for the amount in grams of compound X present at time t minutes.

(6)

(c) Find, according to the model, a general solution for the amount in grams of compound Y present at time t minutes.

(3)

Given that $x = 2$ and $y = 5$ when $t = 0$

(d) find

(i) the particular solution for x,

(ii) the particular solution for y.

(4)

A scientist thinks that the chemical reaction will have stopped after 8 minutes.

(e) Explain whether this is supported by the model.

(1)

a) $\frac{d^2x}{dt^2} = -5\frac{dx}{dt} + 10\frac{dy}{dt}$

$\frac{d^2x}{dt^2} = -5\frac{dx}{dt} + 10\left(-2x + 3y - 4\right)$

$\Rightarrow \frac{dx}{dt} = -5x + 10y - 30$

$\frac{1}{10}\left(\frac{dx}{dt} + 30 + 5x\right) = y$

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Question 5 continued

$$\frac{d^2x}{dt^2} = -5\frac{dx}{dt} + 10\left(-2x + \frac{3}{10}\left(\frac{dx}{dt} + 30 + 5x\right) - 4\right)$$

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} = -20x + 3\frac{dx}{dt} + 90 + 15x - 40$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 50$$ (as required)

b) $m^2 + 2m + 5 = 0$

$m = -1 \pm 2i$

using $m = \alpha \pm \beta i$ in the equation
$x = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$:

$x = e^{-t}(A\cos 2t + B\sin 2t)$

PI : Try $x = K$
$x' = 0$
$x'' = 0$

Sub values in the equation $x'' + 2x' + 5x = 50$:
$0 + 2(0) + 5(K) = 50$
$K = 10$

General Solution : $x = e^{-t}(A\cos 2t + B\sin 2t) + 10$

Question 5 continued

c) $\frac{dx}{dt} = -e^{-t}(A\cos 2t + B\sin 2t) + e^{-t}(2B\cos 2t - 2A\sin 2t)$

Using $\frac{dx}{dt} = -5x + 10y - 30$

$$\frac{\frac{dx}{dt} + 5x + 30}{10} = y$$

$$\frac{1}{10}\Big(\big(-e^{-t}(A\cos 2t + B\sin 2t) + e^{-t}(2B\cos 2t - 2A\sin 2t)\big) + 5\big(e^{-t}(A\cos 2t + B\sin 2t) + 10\big) + 30\Big) = y$$

$$y = \frac{1}{10}e^{-t}\big(4A\cos 2t + 2B\cos 2t + 4B\sin 2t - 2A\sin 2t\big) + 8$$

$$\Rightarrow y = \frac{1}{10}e^{-t}\big((4A+2B)\cos 2t + (4B-2A)\sin 2t\big) + 8$$

di) when $t=0$, $x=2$

$2 = A + 10$

$A = -8$

when $t=0$, $y=5$

$5 = \frac{1}{10}(4A+2B) + 8$

$-3 = \frac{1}{10}(4A+2B)$

$-30 = 4A + 2B$

we know that $A = -8$

$-30 = 4(-8) + 2B$

$-30 = -32 + 2B$

$2 = 2B$

$B = 1$

$\therefore\ x = e^{-t}(\sin 2t - 8\cos 2t) + 10$

dii) $y = e^{-t}(2\sin 2t - 3\cos 2t) + 8$

Question 5 continued

e) when $t = 8$

$x = e^{-8}(\sin 16 - 8\cos 16) + 10 = 9.997512727$

$y = e^{-8}(\sin 16 - 8\cos 16) + 8 = 7.99921753$

when $t \to \infty$

$x \to 10$ as $(e^{-t} \to 0)$

$y \to 8$ as $(e^{-t} \to 0)$

$\therefore$ When $t > 8$, the amount of compound X and the amount of compound Y remain constant at 10 and 8 respectively, which suggests that the chemical reaction has stopped.

This supports the scientist's claim.

(Total for Question 5 is 17 marks)

6. (i) Prove by induction that for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n}(3r+1)(r+2) = n(n+2)(n+3)$$

(6)

(ii) Prove by induction that for all positive **odd** integers n

$$f(n) = 4^n + 5^n + 6^n$$

is divisible by 15

(6)

i) when $n = 1$:

LHS: $(3(1)+1)(1+2) = 12$

RHS: $(1)(1+2)(1+3) = 12$

As LHS = RHS, statement is true for $n = 1$

Assume true for $n = k$:

$$\sum_{r=1}^{k}(3r+1)(r+2) = k(k+2)(k+3)$$

when $n = k+1$

$$\sum_{r=1}^{k+1}(3r+1)(r+2) = (3(k+1)+1)((k+1)+2) + k(k+2)(k+3)$$

$$= (3k+4)(k+3) + k(k+2)(k+3)$$

$$= (k+3)(3k+4 + k(k+2))$$

$$= (k+3)(k^2+5k+4)$$

$$= (k+3)(k+4)(k+1)$$

$$= (k+1)((k+1)+2)((k+1)+3)$$

If the statement is true for $n = k$ then it has been shown true for $n = k+1$ and as it is true for $n = 1$, the statement is true for all positive integers n.

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Question 6 continued

ii) when $n=1$,

$4^1 + 5^1 + 6^1 = 15$

15(1) so the statement is true for $n=1$.

Assume when $n=k$, the statement is divisible by 15.

$f(k) = 4^k + 5^k + 6^k$

when $n=k+2$

$f(k+2) = 4^{k+2} + 5^{k+2} + 6^{k+2}$

$= 16 \times 4^k + 25 \times 5^k + 36 \times 6^k$

$= 16 \times 4^k + 16 \times 5^k + 16 \times 6^k + 9 \times 5^k + 20 \times 6^k$

$= 16f(k) + 45 \times 5^{k-1} + 120 \times 6^{k-1}$

As 15 divides $f(k)$, 45 and 120, so 15 divides $f(k+1)$. If true for $n=k$ then true for $n=k+2$, true for $n=1$ so true for all positive odd integers n.

Question 6 continued

Question 6 continued

(Total for Question 6 is 12 marks)

7. A sample of bacteria in a sealed container is being studied.

The number of bacteria, P, in thousands, is modelled by the differential equation

$$(1+t)\frac{dP}{dt} + P = t^{\frac{1}{2}}(1+t)$$

where t is the time in hours after the start of the study.

Initially, there are exactly 5000 bacteria in the container.

(a) Determine, according to the model, the number of bacteria in the container 8 hours after the start of the study.
(6)

(b) Find, according to the model, the rate of change of the number of bacteria in the container 4 hours after the start of the study.
(4)

(c) State a limitation of the model.
(1)

a) $(1+t)\frac{dP}{dt} + P = t^{\frac{1}{2}}(1+t)$

$\frac{dP}{dt} + \frac{P}{1+t} = t^{\frac{1}{2}}$

$I = e^{\int \frac{1}{1+t} dt} = e^{\ln(1+t)} = 1+t$

$\frac{d}{dt}(1+t)P = t^{\frac{1}{2}}(1+t)$

$(1+t)P = \int t^{\frac{1}{2}}(1+t)\, dt$

$P(1+t) = \int t^{\frac{1}{2}} + t^{\frac{3}{2}}\, dt$

$P(1+t) = \frac{2}{3}t^{\frac{3}{2}} + \frac{2}{5}t^{\frac{5}{2}} + c$

when $t=0$, $P=5$

$5(1+0) = \frac{2}{3}(0)^{\frac{3}{2}} + \frac{2}{5}(0)^{\frac{5}{2}} + c$

$5 = c$

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Question 7 continued

a) when $t = 8$,

$$P = \frac{\frac{2}{3}(8)^{\frac{3}{2}} + \frac{2}{5}(8)^{\frac{5}{2}} + 5}{(1+8)}$$

$= 10.27696434$ (thousands)

$=$ 10,277 bacteria

b) As $P = \dfrac{\frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + 5}{(1+t)}$

$$\Rightarrow \frac{dP}{dt} = \frac{(1+t)(t^{\frac{1}{2}} + t^{3/2}) - \left(\frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2} + 5\right)}{(1+t)^2}$$

when $t = 4$

$$\frac{dP}{dt} = \frac{(1+4)(4^{\frac{1}{2}} + 4^{3/2}) - \left(\frac{2}{3}(4)^{3/2} + \frac{2}{5}(4)^{5/2} + 5\right)}{(1+4)^2}$$

$= \dfrac{403}{375}$ (thousands per hour)

$\dfrac{403}{375} \times 1000 = \dfrac{3224}{3}$

$=$ 1075 bacteria per hour.

Question 7 continued

c) The number of bacteria increases indefinitely, which is not realistic.

Question 7 continued

Question 7 continued

(Total for Question 7 is 11 marks)

TOTAL FOR PAPER IS 75 MARKS